Chaos and Taub-NUT related spacetimes

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Abstract

The occurrence of chaos for test particles moving in a Taub-NUT space-time with a dipolar halo perturbation is studied using Poincaré sections. We find that the NUT parameter (magnetic mass) attenuates the presence of chaos.

PACS numbers: 04.20.Jb, 04.70.Bw, 05.45.+b.

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The Taub-NUT (Newman, Tamburino and Unti) spacetime [1,2] is one of the most bizarre solutions of the vacuum Einstein equations. Because of its many unusual properties it has been described as a "counterexample to almost anything" [3]. It has closed timelike curves, is nonsingular in a meaningful mathematical sense but is not geodesically complete, etc. For certain range of the coordinates, it can be seen as a Schwarzschild monopole endowed with a "magnetic mass" [4].

The Euclidean version of this metric has recently received some attention due to the fact that it is closely related to the dynamics of two non-relativistic Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles [5]. The asymptotic motion of monopoles corresponds to geodesic motion in Euclidean Taub-NUT space, this motion is integrable. This fact has motivated the study of geodesics in Euclidean Taub-NUT and related spaces [6].

Examples of chaotic motion in General Relativity are the geodesic motion of a test particle moving in the geometry associated with : a) Fixed two body problem [7], b) A monopolar center of attraction surrounded by a dipolar halo [8,9] (in Newtonian theory this system is integrable), c) A monopolar center of attraction surrounded by a quadrupole plus octupole halo [10], d) Multi-Curzon and multi-Zipoi-Vorhees solutions [11], and e) A rotating black hole (Kerr geometry) with a dipolar halo [9]. Also gravitational waves can produce irregular motion of test particles orbiting around a static black hole [12,13].

In this Letter we consider the geodesic motion of test particles moving in a Taub-NUT spacetime perturbed by a distant distribution of matter that can be represented by a dipole. The case of a center of attraction (without NUT parameter) perturbed by a dipolar halo was studied in [8], the combined relativistic effects and the breakdown of the reflection symmetry in this case produces a non integrable motion. Our main goal in this letter is to study the effect that the magnetic mass has on the chaotic motion of test particles. The geodesic motion in a pure Taub-NUT spacetime was studied in [14], this case is completely integrable. Also generalizations and perturbations of this spacetime has been considered from the view point of spacetime dynamics [15].

The metric that represents the superposition of a Taub-NUT metric and a dipole along

the z-axis is a stationary axially symmetric spacetime. The vacuum Einstein equations for this class of spacetimes is an integrable system of equations that is closely related to the principal sigma model [16]. Techniques to actually find the solutions are Bäcklund transformations and the inverse scattering method, also a third method constructed with elements of the previous two is the "vesture method", all these methods are closely related [16]. The general metric that represents the nonlinear superposition of a Kerr-NUT solution with a Weyl solution, in particular, with a multipolar expansion can be found by using the "inverse scattering method" [17]. For the particular case of a Taub-NUT metric with a dipolar halo we find

$$ds^2 = g_{tt}(r,z)dt^2 + 2g_{t\phi}(r,z)dtd\phi + g_{\phi\phi}(r,z)d\phi^2 + f(r,z)(dz^2 + dr^2), \tag{1}$$

where

$$\begin{aligned}
g_{tt} &= -e^{-2Duv}[e^{-4Du}(e^{-8Dv}(m+1)^2(u^2-1) + e^{-8D}(m-1)^2(u^2-1) \\
&\quad +2e^{-4(v+1)D}(u^2-v^2)b^2) + e^{-4(v+1)D}(e^{-8Du}+1)(v^2-1)b^2]/H \\
g_{t\varphi} &= -[2e^{-2D(u+v+1)}(e^{-4Du}(e^{-4Dv}(m+1)(u-v)(u+1) \\
&\quad -e^{-4D}(m-1)(u+v)(u-1))(v+1) + e^{-4Dv}(m+1)(u+v) \\
&\quad (u+1)(v-1) - e^{-4D}(m-1)(u-v)(u-1)(v-1))b]/H \\
g_{\varphi\varphi} &= -\sigma^2(e^{-2D(2u-uv)}(e^{-8Dv}(m+1)^2(u+1)^4 + e^{-8D}(m-1)^2(u-1)^4 \\
&\quad +2e^{-4D(v+1)}(u^2-v^2)(u^2-1)b^2)((v^2-1) \\
&\quad +e^{-4D(v+1)}(e^{-8Du}(v+1)^4 + (v-1)^4)(u^2-1)b^2)/H \\
f &= \frac{\sigma^2}{4}H\exp[D^2(u^2v^2 - u^2 - v^2 + 1) + 2D(uv + 2u + 2v + 2)] \\
H &\equiv e^{-4Du}[e^{-8Dv}(m+1)^2(u+1)^2 + e^{-8D}(m-1)^2(u-1)^2 \\
&\quad +2e^{-4D(v+1)}(u^2-v^2)b^2] + e^{-4D(2u+v+1)}(v+1)^2b^2 + e^{-4D(v-1)}(v-1)^2b^2 \\
\sigma^2 &\equiv m^2 + b^2
\end{aligned} \tag{2}$$

The coordinates (r, ϕ, z) are dimensionless and have the range of the usual cylindrical coordinates. They are related to u and v by: $z = uv$ and $r = (u^2-1)^{1/2}(1-v^2)^{1/2}$, $u \geq 1$

and $-1 \leq v \leq -1$. Our units are such that $c = G = 1$; m D and b represent the mass, the halo dipole strength, and b the NUT parameter, respectively. The constant σ represents the geometric "sum" of the "electric" and "magnetic" mass. The coordinate transformation $t' = t$, $u = R/m - 1$, $v = \cos\vartheta$, $\varphi' = \varphi$ reduces (2) with $D = 0$ to the Taub-NUT solution in the usual spherical coordinates [18].

To study the small dipole perturbation case is better to use the metric obtained by keeping the first order terms in the dipolar strength D in the exact metric (2). This approximation, for the parameters and range of coordinates used, will not produce a significant information loss; we shall comeback to this point later. We find for $g_{\mu\nu} = g^0_{\mu\nu} + D g^1_{\mu\nu}$,

$$g_{tt} = -(2(4m^2v - 2m^2 + 2muv + u^2v - 3v + 2)Du - F)(1 - u^2))/F^2$$

$$g_{t\varphi} = 2[-2(((3v^2 - 1)u^2 + (2v - 3)(v + 1)u^4 - v^2 + v)m - 2(u^2 - v^2)m^2u$$

$$+(5v^2 - 2v - 1)u^3 - (3v - 2)uv - u^5)D - F(u^2 - 1)v]b/F^2$$

$$g_{\varphi\varphi} = -\sigma^2[2(2(2((2v^2 - v + 1) - (v + 1)u^2)m^2u + (u^4v + 3u^4 + 6u^2v - 2u^2$$

$$+v - 1)(v - 1)m - (5v - 1)u + (v - 1)u^5 + 4u^3v^2)F(v + 1) - (2((2v + 1)u^2 + v - 1)m$$

$$+(v + 2)u^3 - 2m^2u + 5uv)(4(u^2 + 1)(v^2 - 1)mu - 4(u^2 - v^2)m^2 + 2(3v^2 - 1)u^2$$

$$+(v^2 - 1)u^4 - 3v^2 - 1))(-1)D + (4(u^2 + 1)(v^2 - 1)mu - 4(u^2 - v^2)m^2$$

$$+2(3v^2 - 1)u^2 + (v^2 - 1)u^4 - 3v^2 - 1)F]/F^2$$

$$f = \sigma^2(2(2(u^2 - v + 1)m - (3v - 2)u + 2m^2u + u^3v)D + F)$$

$$F \equiv 1 + 2mu + u^2 \tag{3}$$

The geodesic equations for the metric (1) can be cast as

$$\dot{t} = g^{tb}E_b, \qquad \dot{\phi} = g^{\phi b}E_b, \tag{4}$$

$$\ddot{r} = -\frac{1}{2f}[g^{ab}_{,r}E_aE_b + f_{,r}(\dot{r}^2 - \dot{z}^2) + 2f_{,z}\dot{r}\dot{z}], \tag{5}$$

$$\ddot{z} = -\frac{1}{2f}[g^{ab}_{,z}E_aE_b + f_{,z}(\dot{z}^2 - \dot{r}^2) + 2f_{,r}\dot{r}\dot{z}], \tag{6}$$

where the dots denote derivation with respect to s and the indices a and b take the values (t, ϕ), g^{ab} stands for the inverse of g_{ab}. $E_t = -E$ and $E_\phi = L$ are integration constants;

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E and L are the test particle energy and angular momentum, respectively. The set (4)–(6) admits a third integration constant

$$E_3 = g^{ab} E_a E_b + f(\dot{r}^2 + \dot{z}^2) = -1. \tag{7}$$

Thus to have complete integrability we need one more independent constant of integration. In the case of pure Taub-NUT solution ($D = 0$) we have a fourth constant due to the existence of a third Killing vector and a fifth (Runge-Lenz vector) related to the existence of a Killing-Yano tensor.

The system (5)–(6) can be written as a four dimensional dynamical system in the variables $(r, z, P_r = \dot{r}, P_z = \dot{z})$. A convenient method to study qualitative aspects of this system is to compute the Poincaré sections through the plane $z = 0$. The intersection of the orbits with this plane will be studied in some detail for bounded motions. We shall numerically solve the system (4)–(6) and use the integral (7) to control the accumulated error along the integration; we shall return to this point later.

The Poincaré section for different initial conditions with energy $E = 0.967$ and angular momentum $L = 3.75$ moving in an exact Taub-NUT geometry ($D = 0$) with NUT parameter $b = 0.28$ and "total mass" $\sigma = 1$ are presented in Fig. 1. We have the typical section of an integrable motion, i.e., the sectioning of invariant tori, for integrability and KAM theory, see for instance [19]. The values of σ, E and L will be kept unchanged in our numerical analysis.

The motion of test particles around a Kerr and a Schwarzschild black hole with a dipolar halo ($b = 0$ and $D \neq 0$) is chaotic and it was studied in some detail in [9,8] for different energy shells. In Fig. 2 we show the Poincaré section for $D = 0.0005$ and the same values of $E = 0.967$ and $L = 3.75$ as in Fig. 1. We find islands of integrability surrounded by chaotic motion. The two isolated islands around the points (10, 0.05) and (5, 0.075) correspond to the same torus. In the case studied in [8] they were closer.

Now we shall consider a particle moving around a Taub-NUT attractive center surrounded by a weak dipolar halo. In Fig. 3 we draw the Poincaré section for $D = 0.0005$,

$b = 0.14107$, $\sigma = 1$, $E = 0.967$ and $L = 3.75$ (direct rotation). We see that the islands of stability are larger in this case than in perturbed Schwarzschild solution (see Fig. 2); also we have new systems of small islands immersed in the chaotic region. We have that the chaotic region is smaller in this case than in the perturbed Taub-NUT case. It does look like that the presence of the NUT parameter diminishes the effect of the dipolar strength as a chaos source. In Fig. 4 we present the section with the same parameters of Fig. 3, except that now we have increased the NUT parameter, $b = 0.254787$. In Fig. 4 we observe that the integrable region increases in such a way that the chaotic region has almost disappear in this scale. In other words, the NUT parameter has the property of restoring the invariant tori. In Fig. 5 we present a magnification of the region around the point $(0.0744, 7.37)$ of Fig. 4, we find a small chaotic region around the crossing of the section around the indicated point. Although in the present Letter we present results for particular values of the parameters involved, we did a rather extended numerical study that supports our conclusions, Figs. 3, 4, and 5 being representative of this search. The change of sign of the angular particle angular momentum does not alter the figures and the change of sign of D introduces only a $P_r \rightarrow -P_r$ transformation.

For the values of the parameters $D = 0.0005$, $b = 0.14107, 0.254787$ $E = 0.967$ and $L = \pm 3.75$, we have that the particles move roughly in the "box" $4.7 < r < 21$, $-6 < z < 7$. We take as a measure of error the quantities

$$\Delta g_{ab} = |(g_{ab}^{ex} - g_{ab})/g_{ab}^{ex}|, \quad \Delta f = |(f^{ex} - f)/f^{ex}|, \tag{8}$$

where in these expressions the sum rule of repeated indices does not apply. $g_{\mu\nu}^{ex}$ and $g_{\mu\nu}$ refer to the solutions (2) and (3), respectively. We find that for the above mentioned range of coordinates and the values of parameters used in this Letter the quantities defined in (8) are at most of the order of 10^{-4}. Also in this range, the error in the derivatives of the metric functions is even smaller (the metric functions are very smooth). We also want to mention that the Poincaré sections shown in this Letter were computed with an accumulated error in the "energy" [cf. Eq. (7)] smaller than 10^{-10}.

In summary the NUT parameter has the rather surprising property of making the motion of particles more integrable, it enlarges the region of invariant tori. This is a puzzling result since the addition of the NUT parameter and a dipole moment to a black hole makes the metric rather complicate, hence one should foresee more complex motion with a greater destruction of tori. It happens the opposite. This is again a manifestation of Misner characterization of the Taub-NUT metric as a 'counterexample to almost anything".

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FIGURE CAPTIONS

Fig. 1. Poincaré section of test particles moving with angular momentum $L = 3.75$ and energy $E = 0.967$ in a Taub-NUT geometry with NUT parameter $b = 0.28$ and $\sigma = 1$. This is a typical section of an integrable system.

Fig. 2. Poincaré section for $D = 0.0005$, $E = 0.967$, $L = 3.75$, $b = 0$, and $\sigma = m = 1$. The two isolated islands around the points $(10, 0.05)$ and $(5, 0.075)$ are parts of the same torus.

Fig. 3. Poincaré section for $D = 0.0005$, $b = 0.14107$, $\sigma = 1$ $E = 0.967$ and $L = 3.75$. The islands of stability are larger in this case than in the geometry without NUT parameter (cf. Fig. 2).

Fig. 4. Poincaré section with the same parameters of Fig. 3, except that now $b = 0.254787$. The chaotic region is almost inexistent.

Fig. 5. The region around the point $(0.0744, 7.37)$ of Fig. 4 is magnified. We find a small chaotic region.